Due Sun

$A \subset B$ $A \subseteq B$ $x \leq 3$ $x < 3$

4.9 – Rank, Nullity, and the Fundamental Matrix Spaces

vector space spanned by row vectors
column

$A = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$ 3×6

$col(A) \subseteq R^3$ $row(A) \subseteq R^6$

Theorem 4.9.1 The row space and column space of a matrix A have the same dimension.

same # of basis vectors, based on pivots

\# of pivots

Definition: The common dimension of the row space and column space of a matrix A is called the **rank** of A and is denoted by rank(A); the dimension of the null space of A is called the **nullity** of A and is denoted by nullity(A). That is,

$$\text{rank}(A) = \dim\,[\text{row}\,(A)] = \dim\,[\text{col}\,(A)] \text{ and } \text{nullity}(A) = \dim\,[\text{null}\,(A)].$$

set of solutions to homog. system

a.k.a. dim(kernel) of T_A

#1 Find the rank and nullity of the matrix A by reducing it to row echelon form.

a. $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\boxed{\begin{array}{l} \text{rank}(A) = 1 \\ \text{nullity}(A) = 3 \end{array}}$ (# of pivots)
(# of free variables)

soln: $x_1 = -2x_2 + x_3 - x_4 \Rightarrow$

$$\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} t$$

Basis has 3 vectors.

b. $A = \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & -1 & -2 & -6 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

rank(A) = 3 (# pivots)

nullity(A) = 2 (# parameters)

Theorem 4.9.2 Dimension Theorem for Matrices
If A is a matrix with n columns, then rank(A) + nullity$(A) = n$.

total # variables

leading variables (pivots)

free variables (parameters)

Theorem 4.9.3 If A is an $m \times n$ matrix, then
a) rank(A) = the number of leading variables in the general solution of $A\mathbf{x} = \mathbf{0}$.
b) nullity(A) = the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$.

Theorem 4.9.4 If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system of m equations in n unknkowns, and if A has rank r, then the general solution of the system contains $n - r$ parameters.

Rearrangement of dimension theorem.

Fundamental spaces of a matrix

Considering a matrix A and its transpose A^T, we have

$$\begin{tikzcd}
\text{row}(A) \arrow[dr, "\text{Same}"] & \text{row}(A^T) \\
\text{col}(A) \arrow[ur, "\text{Same}"'] & \text{col}(A^T) \\
\text{null}(A) & \text{null}(A^T)
\end{tikzcd}$$

The four **fundamental spaces** of a matrix A are row (A), col (A), null (A), and null $\left(A^T\right)$. The null space of A^T is also called the **left null space of A**. The **left nullity of A** is nullity $\left(A^T\right) = \dim\left[\text{null}\left(A^T\right)\right]$.

Because $A^T\vec{x}=\vec{0}$ becomes $\vec{x}^TA=\vec{0}^T$

Theorem 4.9.5 If A is any matrix, then rank(A) = rank $\left(A^T\right)$.

(# Pivots doesn't change)

Let A be an $m \times n$ matrix

Since $\underline{\text{rank}(A^T)} + \text{nullity}(A^T) = m$,

Thm 4.9.5 yields $\underline{\text{rank}(A)} + \text{nullity}(A^T) = m$

So

$$\dim[\text{row}(A)] = \underline{r} \qquad \dim[\text{col}(A)] = \underline{r}$$

$$\dim[\text{null}(A)] = n-\underline{r} \qquad \dim[\text{null}(A^T)] = m-\underline{r}$$

To find bases for the 4 fundamental spaces of an $m \times n$ matrix A, form

$$[\underline{A} \mid I_m] \xrightarrow{\text{rref}} [\underline{R} \mid \underline{E}]$$

- A basis for $\underline{\text{row}(A)}$ is the r rows of $\underline{R}$ that contain leading 1s (pivot rows)

- A basis for $\underline{\text{col}(A)}$ is the r columns of $\underline{A}$ that correspond to pivot columns of R

- A basis for null(A) is found in the general solution of $A\vec{x} = \vec{0}$.

- A basis for $\underline{\text{null}(A^T)}$ - a.k.a. the left null space basis - is the bottom $m-r$ rows of $\underline{E}$.

(pf is at the end of this section)

m×n

#11 Find the dimensions and bases for the four fundamental spaces of the matrix.

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \\ -9 & 0 \end{bmatrix} \quad \left[\begin{array}{cc|ccc} 1 & 4 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ -9 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|ccc} 1 & 0 & 1 & -4/3 & 0 \\ 0 & 1 & 0 & 1/3 & 0 \\ 0 & 0 & 9 & -12 & 1 \end{array}\right]$$

row(A) basis: $\{[1 \ \ 0], [0 \ \ 1]\}$

col(A) basis: $\left\{\begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}\right\}$

$n=2,\ r=2 \Rightarrow n-r = 0 = \dim[\text{null}(A)]$

$= \text{nullity}(A)$

null(A) basis: $\emptyset = \{ \ \}$

null(A^T): 3 rows, rank is 2 $\Rightarrow$ use

bottom 3-2 rows of E

null(A^T) basis: $\{[9 \ -12 \ \ 1]\}$.

Recall (Thm 3.4.3, which we saw in 3.3)

that if A is an m×n matrix, then the

solution set of the homogeneous system

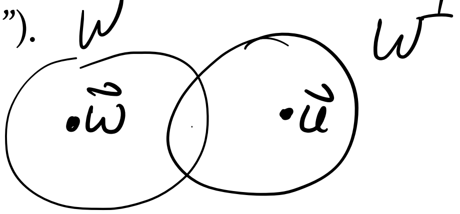
$A\vec{x}=\vec{0}$ consists of all vectors in $\mathbb{R}^n$ that

are orthogonal to every row vector of A.

The set of solutions $\{\vec{x} \mid A\vec{x}=\vec{0}\}$ is the

orthogonal complement of row(A).

Definition: If W is a subspace of R^n, then the set of all vectors in R^n that are orthogonal to every vector in W is called the **orthogonal complement** of W and is denoted by the symbol $W^\perp$ (pronounced "W perp").



Theorem 4.9.6 If W is a subspace of R^n, then:
a) $W^\perp$ is a subspace of R^n.
b) The only vector common to W and $W^\perp$ is **0**, that is, $W \cap W^\perp = \{\mathbf{0}\}$. c) The orthogonal complement of $W^\perp$ is W, that is, $(W^\perp)^\perp = W$.

Text: If you see $W_\perp$, it means $W^\perp$.

Pf: a) $\vec{0} \perp \vec{v}$ for all $\vec{v} \in R^n$,
so $\vec{0} \perp \vec{w} \in W \Rightarrow W^\perp$ is not empty.
Let $\vec{w}_1, \vec{w}_2 \in W^\perp$. Then $\vec{w}_1 \cdot \vec{w} = 0$
and $\vec{w}_2 \cdot \vec{w} = 0$ $\forall \vec{w} \in W$ and $\vec{w} \in W$.
$(\vec{w}_1 + \vec{w}_2) \cdot \vec{w} = \vec{w}_1 \cdot \vec{w} + \vec{w}_2 \cdot \vec{w}$
$= 0 + 0 = 0$
$\vec{w}_1 + \vec{w}_2 \in W^\perp$. And if $k \in R$,
$(k\vec{w}_1) \cdot \vec{w} = k(\vec{w}_1 \cdot \vec{w}) = k(0) = 0 \Rightarrow k\vec{w}_1 \in W^\perp$.
$W^\perp$ is a subspace of R^n.

b) Let $\vec{w} \in W \cap W^\perp$. Then $\vec{w} \in W$ and
$\vec{w} \in W^\perp \Rightarrow \vec{w} \cdot \vec{w} = 0 \Rightarrow \vec{w} = \vec{0}$ by
the positivity property.
Thus, $W \cap W^\perp = \{\vec{0}\}$.

Theorem 4.9.7 If A is an $m \times n$ matrix, then:
a) The null space of A and the row space of A are orthogonal complements in R^n.
b) The null space of A^T and the column space of A are orthogonal complements in R^m.

#15 Confirm the orthogonality statements in the two parts of Theorem 4.9.7 for the matrix $A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \\ -9 & 0 \end{bmatrix}$.

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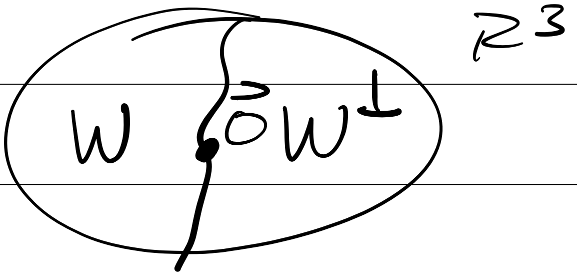
a) null(A) basis is $\emptyset$, row(A) basis: $\{[1\ 0], [0\ 1]\}$.
null(A) $= \{\vec{0}\}$, row(A) $=$ span$\{[1\ 0], [0\ 1]\}$.

✓

b) null$(A^T) =$ span$\left\{ \underbrace{\begin{bmatrix} 9 \\ -12 \\ 1 \end{bmatrix}}_{\vec{v}_1} \right\}$; col(A) $=$ span$\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix}}_{\vec{v}_2}, \underbrace{\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}}_{\vec{v}_3} \right\}$

$\vec{v}_1 \cdot \vec{v}_2 = 9 - 9 = 0$; $\vec{v}_1 \cdot \vec{v}_3 = 36 - 36 = 0$

Since the basis vectors in each set are orthogonal, the spans are orthogonal complements in R^3. (See HW for this section)



#27 Suppose that A is a 3×3 matrix whose null space is a line through the origin in 3-space. Can the row or column space of A also be a line through the origin? Explain.

R^3: dimension is 3

line through origin: dimension is 1

No. row(A) is $(\text{null}(A))^{\perp}$, so it must have dimension 2. If nullity (A) = 1, then rank(A) = 2. It is a plane through the origin

Theorem 4.9.8 Equivalent Statements (extends Theorem 2.3.8)

If A is an $n \times n$ matrix, then the following statements are equivalent.

a) A is invertible.

b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

c) The reduced row echelon form of A is I_n.

d) A is expressible as a product of elementary matrices.

e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

g) $\det(A) \neq 0$.

h) The column vectors of A are distinct and linearly independent.

i) The row vectors of A are distinct and linearly independent.

j) The column vectors of A span R^n.

k) The row vectors of A span R^n.

l) The column vectors of A form a basis for R^n.

m) The row vectors of A form a basis for R^n.

n) A has rank n.

o) A has nullity 0.

p) The orthogonal complement of the null space of A is R^n.

q) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.

Pf: b $\Rightarrow$ h: $A\vec{x}$ is a linear combination of column vectors of A. $A\vec{x} = \vec{0}$ has only the trivial solution so the column vectors are lin. indep.

h $\Rightarrow$ j, l, m: The n column vectors of A are a linearly independent set in R^n $\Rightarrow$ they span R^n and so are a basis for R^n. Since the n column vectors are a basis for col(A), rank(A) = n.